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ON SOME NEW CLASSES OF COMPACT-LIKE BITOPOLOGICAL SPACES

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Abstract. In this paper, we have introduced a new type of covering property $\beta^t_{(\omega_r,s)}$-closedness, stronger than $P^t_{(\omega_r,s)}$-closedness [3] in terms of (r,s)-β-open sets [9] and β-ω_t-closures in bitopological spaces along with its several characterizations via filter bases and grills [15] and various properties. Further grill generalizations of $\beta^t_{(\omega_r,s)}$-closedness (namely, $\beta^t_{(\omega_r,s)}$-closedness modulo grill) and associated concepts have also been investigated.

1. Introduction

In the year 1963, J. C. Kelly [8] has introduced the concept of bitopological spaces which now becomes a mature field of topology. The notion of ω-open sets introduced by H. Z. Hdeib [5, 6] has been studied by a good number of researchers in recent times, namely Noiri, Omari and Noorani [11, 12], Omari and Noorani [13, 14], Zoubi and Nashef [16] and Afsan and Basu [2, 3, 4]. In the present paper, in section 3, we have initiated β-(ω_r,s)-θ_t-closures of sets and allied concepts in bitopological spaces which have been exploited effectively in investigating certain concepts which have been developed in the subsequent sections. In section 4, a new type of covering property $\beta^t_{(\omega_r,s)}$-closedness is introduced in a bitopological space (X,τ_1,τ_2) using (r,s)-β-open sets [9] and β-ω_t-closures. Several characterizations via filter bases and grills [15] and various properties of such concept have been discussed. Further, in the last section, the concept of $\beta^t_{(\omega_r,s)}$-closedness modulo grill along with its simple properties have also been investigated.

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2. Prerequisites

Let (X, τ) be a topological space and $A \subset X$. Then a point $x \in X$ is called a condensation point of A if for each open set U containing x, $A \cap U$ is uncountable. A set A is called ω-closed [5] if it contains all of its condensation points and the complement of an ω-closed set is called an ω-open set [5] or equivalently, $A \subset X$ is ω-open if and only if for each $x \in A$ there exists an open set U containing x such that $U - A$ is countable. Throughout this paper, spaces (X, τ_1, τ_2) and $((Y, \varrho_1, \varrho_2)$ (or simply X and Y) represent non-empty bitopological spaces and r, s and t are indices varying over the set $\{1, 2\}$. The set of all ω-open sets of space X when the topology τ_r is considered is denoted by τ_{ω_r}. It is to be noted that τ_{ω_r} is a topology on X finer than τ_r [5]. A subset A of a topological space X is called β-open [1] (resp. pre-ω-open [11], β-ω-open [11]) if $A \subset cl(int(cl(A)))$ (resp. $A \subset int_\omega(cl(A))$ and $A \subset cl(int_\omega(cl(A))))$. The closure (resp. interior, ω-interior [11], ω-closure [11], pre-ω-interior [4], pre-ω-closure [4]) of a subset A of a space X with respect to the topology τ_r (where $r = 1, 2$) (read as r-closure (resp. r-interior, ω_r-interior, ω_r-closure, pre-ω_r-interior, pre-ω_r-closure, r-β-closure)) are denoted by $cl_r(A)$ (resp. $int_r(A)$, $int_{\omega_r}(A)$, $cl_{\omega_r}(A)$, $pint_{\omega_r}(A)$, $pcl_{\omega_r}(A)$, $\beta cl_r(A)$). A subset A of a bitopological space X is called (resp. (r, s)-preopen [7], and (r, s)-β-open [9]) if $A \subset int_r(cl_s(A))$ (resp. $A \subset cl_s(int_r(cl_s(A))))$. The complement of (r, s)-preopen (resp. (r, s)-β-open) set is called (r, s)-preclosed [7] (resp. (r, s)-β-closed [9]). The family of all (r, s)-preopen (resp. (r, s)-β-open, and (r, s)-β-clopen) subsets of X is denoted by (r, s)-$PO(X)$ (resp. (r, s)-$\beta O(X)$ and (r, s)-$\beta CO(X)$). The family of all (r, s)-preopen (resp. (r, s)-β-open, and (r, s)-β-clopen) subsets of X containing a point x of X is denoted by (r, s)-$PO(X, x)$ (resp. (r, s)-$\beta O(X, x)$ and (r, s)-$\beta CO(X, x)$) and the family of all (r, s)-β-open subsets of X containing a subset $K \subset X$ is denoted by (r, s)-$\beta O(X, K)$. The family of all β-open sets in the topological space (X, τ_r) is denoted by $(r)\beta O(X)$. $(r, s)\beta cl(A)$ is the intersection of all (r, s)-β-closed subsets of X containing A [9].

Thron [15] has defined a grill as a non-empty family $\mathcal{G}$ of non-empty subsets of X satisfying (a) $A \in \mathcal{G}$ and $A \subset B \Rightarrow B \in \mathcal{G}$ and (b) $A \cup B \in \mathcal{G} \Rightarrow$ either $A \in \mathcal{G}$ or $B \in \mathcal{G}$. Thron [15] also has shown that $\mathcal{F}(\mathcal{G}) = \{A \subset X : A \cap F \neq \emptyset, \forall F \in \mathcal{G}\}$ is a filter on X and there exists an ultrafilter $\mathcal{F}$ such that $\mathcal{F}(\mathcal{G}) \subset \mathcal{F} \subset \mathcal{G}$.

3. β-(ω_r, s)-θ_t-closure operators

Definition 3.1. Let A be any subset of a topological space X. Then β-ω-interior (resp. β-ω-closure) of A is the set $\beta int^{\tau}_{\omega}(A) = \cup\{S \subset A : S \in \beta\omega O(X)\}$ (resp. $\beta cl^{\tau}_{\omega}(A) = \cap\{S \supset A : X - G \in \beta\omega O(X)\}$). If no confusion arises, β-ω-interior (resp. β-ω-closure) of A is denoted by $\beta int_{\omega}(A)$ (resp. $\beta cl_{\omega}(A)$). If X is a bitopological space, then the β-ω-interior (resp. β-ω-closure) of A with respect to the topology τ_t is denoted by $\beta int_{\omega_t}(A)$ (resp. $\beta cl_{\omega_t}(A)$).

Now we state the following theorem.

Theorem 3.2. *Let A and B be any two subsets of a topological space X. Then the following properties hold:*
(a) $\beta cl_{\omega}(A) \subset \beta cl(A)$, $\beta cl_{\omega}(A) \subset pcl_{\omega}(A)$ and $\beta cl_{\omega}(A) \subset cl_{\omega}(A)$.
(b) $A \subset B$ implies $\beta cl_{\omega}(A) \subset \beta cl_{\omega}(B)$ and $\beta int_{\omega}(A) \subset \beta int_{\omega}(B)$.
(c) $\beta cl_{\omega}(\beta cl_{\omega}(A)) = \beta cl_{\omega}(A)$ and $\beta int_{\omega}(\beta int_{\omega}(A)) = \beta int_{\omega}(A)$.
(d) A is β-ω-closed if and only if $\beta cl_{\omega}(A) = A$.
(e) A is β-ω-open if and only if $\beta int_{\omega}(A) = A$.
(f) $\beta cl_{\omega}(X - A) = X - \beta int_{\omega}(A)$.
(g) $\beta int_{\omega}(X - A) = X - \beta cl_{\omega}(A)$.

Remark 3.3. For a subset A of a topological space, $\beta cl_{\omega}(A) \neq \beta cl(A)$ and $\beta cl_{\omega}(A) \neq pcl_{\omega}(A)$ in general, which is shown in the following examples.

Example 3.4. *(a) Consider the space $X = \mathbf{Q}$ with the topology generated by the base $\mathcal{B} = \{B_r : r \in \mathbf{Q}\}$, where $B_r = \{1, r\}$. Then the topology on X is $\tau = \{\emptyset\} \cup \{S \subset \mathbf{Q} : S$ contains $1\} = \beta O(X)$ and $\tau_{\omega} = P(X) = \beta\omega O(X)$, where $P(X)$ is the power set of X. Let A be the set of all odd integers. Then $\beta cl_{\omega}(A) = A$ and $\beta cl(A) = \mathbf{Q}$.*
(b) Consider the real line $X = \mathbf{R}$ with the usual topology τ. Then $\tau = \tau_{\omega}$ and so $(0, 1] \in \beta\omega O(X) - P\omega O(X)$. Then $\beta cl_{\omega}((-\infty, 0] \cup (1, \infty)) = (-\infty, 0] \cup (1, \infty)$ and $pcl_{\omega}((-\infty, 0] \cup (1, \infty)) = (-\infty, 0] \cup [1, \infty)$.

Definition 3.5. A point $x \in X$ is said to be a β-(ω_r, s)-θ_t-accumulation (resp. (r, s)-θ_t-β-accumulation) point of a subset A of a bitopological space X if $\beta cl_{\omega_t}(U) \cap A \neq \emptyset$ (resp. $\beta cl_t(U) \cap A \neq \emptyset$) for every $U \in (r, s)\beta O(X, x)$. The set of all β-(ω_r, s)-θ_t-accumulation (resp. (r, s)-θ_t-β-accumulation) points of A is called the β-(ω_r, s)-θ_t-closure (resp. (r, s)-θ_t-β-closure) of A and is denoted by $\beta_{(\omega_r, s)} cl_{\theta_t}(A)$ (resp. $(r, s)\beta cl_{\theta_t}(A)$). A subset A of a bitopological space X is said to be β-(ω_r, s)-θ_t-closed (resp. $(r, s)\theta_t$-β-closed set) if $\beta_{(\omega_r, s)} cl_{\theta_t}(A) = A$ (resp.

$(r,s)\beta cl_{\theta_t}(A) = A$). The complement of a β-(ω_r, s)-θ_t-closed set (resp. $(r,s)\theta_t$-β-closed set) is called β-(ω_r, s)-θ_t-open set (resp. $(r,s)\theta_t$-β-open set).

If $\tau_1 = \tau_2 = \tau$, i.e. if X is a topological space, we drop the indices r, s and t in the the terms of the definitions.

LEMMA 3.6. *A subset A of space X is a β-(ω_r, s)-θ_t-open if and only if for each $x \in A$, there exists $V \in (r,s)\beta O(X,x)$ such that $\beta cl_{\omega_t}(V) \subset A$.*

Proof. Let A be a β-(ω_r, s)-θ_t-open set. Suppose $x \in A$. Then $X - A$ is β-(ω_r, s)-θ_t-closed and thus for each $x \in A$, there exists a $V \in (r,s)\beta O(X,x)$ such that $\beta cl_{\omega_t}(V) \cap (X - A) = \emptyset$. Hence $\beta cl_{\omega_t}(V) \subset A$.

Conversely, let there exists an $x \in A$ such that $\beta cl_{\omega_t}(V) \not\subset A$ for all $V \in (r,s)\beta O(X,x)$. Thus $\beta cl_{\omega_t}(V) \cap (X - A) \neq \emptyset$ for all $V \in (r,s)\beta O(X,x)$ and so $x \in \beta_{(\omega_r,s)}cl_{\theta_t}(X - A)$. Hence $X - A$ is not β-(ω_r, s)-θ_t-closed. □

Now we state following theorem:

THEOREM 3.7. *Let A and B be any subsets of a bitopological space X. Then the following properties hold:*
(a) *$(r,s)\theta_t$-β-closed sets and pre-(ω_r, s)-θ_t-closed sets are β-(ω_r, s)-θ_t-closed sets.*
(b) *$\beta_{(\omega_r,s)}cl_{\theta_t}(A) \subset (r,s)\beta cl_{\theta_t}(A)$ and $\beta_{(\omega_r,s)}cl_{\theta_t}(A) \subset p_{(\omega_r,s)}cl_{\theta_t}(A)$,*
(c) *$A \subset B$ implies $\beta_{(\omega_r,s)}cl_{\theta_t}(A) \subset \beta_{(\omega_r,s)}cl_{\theta_t}(B)$,*
(d) *the intersection of an arbitrary family of β-(ω_r, s)-θ_t-closed sets is β-(ω_r, s)-θ_t-closed in X.*

Proof. Proof of (a), (b), (c) are straightforward. So we prove (d) only.

(d): Let $\{A_\alpha : \alpha \in \Delta\}$ be a family of β-(ω_r, s)-θ_t-closed sets. Let $x \in \beta_{(\omega_r,s)}cl_{\theta_t}(\cap_{\alpha\in\Delta}(A_\alpha))$. Then for all $U \in (r,s)\beta O(X,x)$, $\emptyset \neq (\cap_{\alpha\in\Delta}(A_\alpha)) \cap \beta cl_{\omega_t}(U) = \cap_{\alpha\in\Delta}(A_\alpha \cap \beta cl_{\omega_t}(U))$. So for each $\alpha \in \Delta$, $A_\alpha \cap \beta cl_{\omega_t}(U) \neq \emptyset$. Thus $x \in \beta_{(\omega_r,s)}cl_{\theta_t}(A_\alpha) = A_\alpha$ for each $\alpha \in \Delta$ and hence $x \in \cap_{\alpha\in\Delta}(A_\alpha)$. Thus $\cap_{\alpha\in\Delta}(A_\alpha)$ is β-(ω_r, s)-θ_t-closed in X.

The following examples reflect the fact that the converse of Theorem 3.7.(b) is not true, i.e. $\beta_{(\omega_r,s)}cl_{\theta_t}(A) \neq (r,s)\beta cl_{\theta_t}(A)$ and $\beta_{(\omega_r,s)}cl_{\theta_t}(A) \neq p_{(\omega_r,s)}cl_{\theta_t}(A)$. In fact $\beta_\omega cl_\theta(A) \neq \beta cl_\theta(A)$ and $\beta_\omega cl_\theta(A) \neq p_\omega cl_\theta(A)$. □

EXAMPLE 3.8. *(a) Consider the topological space $X = \mathbf{Q}$ with the topology τ as defined in Example 3.4.(a) and Let A be the set of all odd integers. Then $\beta_\omega cl_\theta(A) = A$ and $\beta cl_\theta(A) = \mathbf{Q}$.*

(b) Consider the real line $X = \mathbf{R}$ with the usual topology τ and $A = (-\infty, 0] \cup (1, \infty)$. Then $1 \in p_\omega cl_\theta(A) - \beta_\omega cl_\theta(A)$.

4. $\beta^t_{(\omega_r,s)}$-closed spaces

Definition 4.1. A subset S of a bitopological space X is called $\beta^t_{(\omega_r,s)}$-closed (resp. $\beta^t_{(r,s)}$-closed) relative to X if every (r,s)-β-open cover of X has a finite subfamily whose β-ω_t-closures (resp. t-β-closures) cover S. If $S = X$, then the $\beta^t_{(\omega_r,s)}$-closed (resp. $\beta^t_{(r,s)}$-closed) set S relative to X is called $\beta^t_{(\omega_r,s)}$-closed (resp. $\beta^t_{(r,s)}$-closed) bitopological space. We read $\beta^t_{(\omega_r,s)}$-closedness as β_ω-closedness if the topologies on X are taken the same.

It is easy to verify that every $\beta^t_{(\omega_r,s)}$-closed subset of any bitopological space is $\beta^t_{(r,s)}$-closed and $P^t_{(\omega_r,s)}$-closed. Again in a But a $\beta^t_{(r,s)}$-closed subset of any bitopological space need not be $\beta^t_{(\omega_r,s)}$-closed.

Example 4.2. *Consider the space $X = \mathbf{Q}$ with the topology $\tau_1 = \{\emptyset\} \cup \{S \subset \mathbf{Q} : S \text{ contains } 1\}$ and $\tau_2 = \{\emptyset, X\}$. Then $(2,1)\beta O(X) = \{\emptyset\} \cup \{S \subset \mathbf{Q} : S \text{ contains } 1\}$ and $\tau_{\omega_1} = P(X) = (1)\beta\omega O(X)$, where $P(X)$ is the power set of X. So, X is $\beta^1_{(2,1)}$-closed because every $(2,1)$-β-open set has $\mathbf{Q}$ as its (1)-β-closure. but since the $(2,1)$-β-open cover $\{\{1,r\} : r \in \mathbf{Q}\}$ has no finite subfamily whose β-ω_1-closures cover S. So X is not $\beta^1_{(\omega_2,1)}$-closed.*

Theorem 4.3. *Let A and B be subsets of X. If A is β-(ω_r,s)-θ_t-closed and B is $\beta^t_{(\omega_r,s)}$-closed relative to X, then $A \cap B$ is $\beta^t_{(\omega_r,s)}$-closed relative to X.*

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be a cover of $A \cap B$ by (r,s)-β-open subsets of X. Since A is β-(ω_r,s)-θ_t-closed, then for each $x \in B - A$ there exists $V_x \in (r,s)$-$\beta O(X,x)$ such that $\beta cl_{\omega_t}(V_x) \cap A = \emptyset$. Then $\{U_\alpha : \alpha \in \Delta\} \cup \{V_x : x \in B - A\}$ is a cover of B by (r,s)-β-open subsets of X. So the $\beta^t_{(\omega_r,s)}$-closedness of B relative to X, ensures there existence of finite number of points $x_1, x_2, ..., x_n \in B - A$ and finite number of indices $\alpha_1, \alpha_2, ..., \alpha_m \in \Delta$ such that $B \subset (\cup_{i=1}^n \beta cl_{\omega_t}(V_{x_i})) \cup (\cup_{j=1}^m \beta cl_{\omega_t}(U_{\alpha_j}))$ and thus $A \cap B \subset \cup_{j=1}^m \beta cl_{\omega_t}(U_{\alpha_j})$. So $A \cap B$ is $\beta^t_{(\omega_r,s)}$-closed relative to X. □

Corollary 4.4. *If X is a $\beta^t_{(\omega_r,s)}$-closed bitopological space (resp. β_ω-closed topological space), then every β-(ω_r,s)-θ_t-closed (resp. β-ω-θ-closed) subset of X is $\beta^t_{(\omega_r,s)}$-closed (resp. β_ω-closed) relative to X.*

Theorem 4.5. *A β_ω-closed set S relative to a T_2 topological space X is β-ω-θ-closed if S is β-ω-open.*

Proof. Let $x \in X - S$. Then for each $p \in S$, there exist two disjoint open sets U_p and V_p such that $x \in U_p$ and $p \in V_p$. Then $\{V_p : p \in S\}$ is a cover of S by open (and so β-open) sets of X. Since S is β_ω-closed set relative to X, there exist $p_1, p_2, \ldots\ldots., p_n \in S$ such that $S \subset \cup_{i=1}^n \beta cl_\omega(V_{p_i}) \subset \beta cl_\omega(\cup_{i=1}^n (V_{p_i}))$. Now consider $U = \cap U_{p_i}$ and $V = \cup V_{p_i}$. Then $U \cap V = \emptyset$ and so $\beta cl_\omega(V) \subset cl_\omega(V) \subset cl(V) \subset X - U$. Hence $\beta cl_\omega(V) \cap U = \emptyset$ and so $\beta int_\omega(\beta cl_\omega(V)) \cap U = \emptyset$. Thus $\beta int_\omega(\beta cl_\omega(V)) \cap \beta cl_\omega(U) = \emptyset$. But since S is β-ω-open, $S \cap \beta cl_\omega(U) = \emptyset$. Therefore S is β-ω-θ-closed. □

Definition 4.6. A filter base $\mathcal{F}$ (resp. a grill $\mathcal{G}$) on a topological space (X, τ_1, τ_2) is said to β-(ω_r, s)-θ_t-converge to a point $x \in X$ if for each $V \in (r,s)\beta O(X,x)$, there exists $F \in \mathcal{F}$ (resp. $F \in \mathcal{G}$) such that $F \subset \beta cl_{\omega_t}(V)$. A filter base $\mathcal{F}$ is said to β-(ω_r, s)-θ_t-accumulate (or β-(ω_r, s)-θ_t-adhere) at $x \in X$ if $\beta cl_{\omega_t}(V) \cap F \neq \emptyset$ for every $V \in (r,s)\beta O(X,x)$ and every $F \in \mathcal{F}$. The collection of all the points of X at which the filter base $\mathcal{F}$ β-(ω_r, s)-θ_t-adheres is denoted by $\beta_{(\omega_r,s)}$-θ_t-$ad\mathcal{F}$. We read the term β-(ω_r, s)-θ_t-convergence (resp. β-(ω_r, s)-θ_t-adherent point) as β-ω-θ-convergence (resp. β-ω-θ-adherent point) if the topologies on X are taken the same.

Theorem 4.7. *An ultrafilter base $\mathcal{F}$ β-(ω_r, s)-θ_t-converges to a point $x \in X$ if and only if it β-(ω_r, s)-θ_t-accumulates to the point x.*

Proof. Here only to prove is that if $\mathcal{F}$ β-(ω_r, s)-θ_t-accumulates to the point x, then $\mathcal{F}$ β-(ω_r, s)-θ_t-converges to the point x. If $\mathcal{F}$ does not β-(ω_r, s)-θ_t-converge to the point x, there exists a $V \in (r,s)\beta O(X,x)$ such that $F \not\subset \beta cl_{\omega_t}(V)$ and so $(X - \beta cl_{\omega_t}(V)) \cap F \neq \emptyset$ for every $F \in \mathcal{F}$. Since $\mathcal{F}$ is an ultrafilter base on X, $\beta cl_{\omega_t}(V) \in \mathcal{F}$. Again since $\mathcal{F}$ β-(ω_r, s)-θ_t-accumulates to the point x, $\beta cl_{\omega_t}(V) \cap F \neq \emptyset$ for every $F \in \mathcal{F}$ and so $\beta cl_{\omega_t}(V) \in \mathcal{F}$. □

Theorem 4.8. *For a bitopological space X the following conditions are equivalent:*
(a) X is $\beta^t_{(\omega_r,s)}$-closed,
(b) for every family $\{V_\alpha : \alpha \in \Delta\}$ of (r,s)-β-closed subsets such that

$\cap\{V_\alpha : \alpha \in \Delta\} = \emptyset$, *there exist* $\alpha_1, \alpha_2, ..., \alpha_n \in \Delta$ *such that* $\cap_{i=1}^n \beta int_{\omega_t}(V_{\alpha_i}) = \emptyset$,
(c) every ultrafilter base β-(ω_r, s)-θ_t-*converges to some point of* X,
(d) every filter base β-(ω_r, s)-θ_t-*adheres at some point of* X,
(e) every grill on X β-(ω_r, s)-θ_t-*converges to some point of* X.

Proof. **(a)⇔(b).** Consider $\Sigma = \{V_\alpha : \alpha \in \Delta\}$ be a family of (r, s)-β-closed subsets such that $\cap\{V_\alpha : \alpha \in \Delta\} = \emptyset$. Then there exist finite number of indices $\alpha_1, \alpha_2, ..., \alpha_k \in \Delta$ such that $\cup_{i=1}^n \beta cl_{\omega_t}(X - V_{\alpha_i}) = X$. Hence $X - \cup_{i=1}^n \beta cl_{\omega_t}(X - V_{\alpha_i}) = \emptyset$ and so by Theorem 3.2, $\cap_{i=1}^n \beta int_{\omega_t}(V_{\alpha_i}) = \emptyset$.

Conversely, let $\{U_\alpha : \alpha \in \Delta\}$ be an (r, s)-β-open cover of X. Then $\{X - U_\alpha : \alpha \in \Delta\}$ is a family of (r, s)-β-closed subsets of X with $\cap\{X - U_\alpha : \alpha \in \Delta\} = \emptyset$. Thus by (b), there exist $\alpha_1, \alpha_2, ..., \alpha_n \in \Delta$ such that $\cap_{i=1}^n \beta int_{\omega_t}(X - U_{\alpha_i}) = \emptyset$ i.e. by Theorem 3.2, $\cup_{i=1}^n \beta cl_{\omega_t}(U_{\alpha_i}) = X$. So X is $\beta^t_{(\omega_r,s)}$-closed.

(b)⇒(c). Let $\mathcal{F}$ be an ultrafilter base on X which does not β-(ω_r, s)-θ_t-converge to any point of X. Then by Theorem 4.7, $\mathcal{F}$ can not β-ω-θ_t-accumulate at any point of X and so for each $x \in X$, there are an $F_x \in \mathcal{F}$ and a $V_x \in (r, s)\beta O(X, x)$ such that $\beta cl_{\omega_t}(V_x) \cap F_x = \emptyset$ and so $F_x \subset X - \beta cl_{\omega_t}(V_x) = \beta int_{\omega_t}(V_x)$ (by Theorem 3.2). Again $\{X - V_x : x \in X\}$ is a family of (r, s)-β-closed subsets of X satisfying $\cap\{X - V_x : x \in X\} = \emptyset$. So there exists a finite subset X_0 of X such that $\cap_{x\in X_0} \beta int_{\omega_t}(V_x) = \emptyset$. Since $\mathcal{F}$ is a filter base on X, there exists $F_0 \in \mathcal{F}$ such that $F_0 \subset \cap_{x\in X_0} F_x$ and so $F_0 = \emptyset$ — a contradiction.

(c)⇒(d). Let $\mathcal{F}$ be any filter base on X and $\mathcal{F}_0$ be an ultrafilter base on X containing $\mathcal{F}$. Then (c) ensures that $\mathcal{F}_0$ β-(ω_r, s)-θ_t-converges to some point $x \in X$. Therefore for each $V \in (r, s)\beta O(X, x)$, there exists $F_0 \in \mathcal{F}_0$ such that $F_0 \subset \beta cl_{\omega_t}(V)$. Now since for each $F \in \mathcal{F}$, $\emptyset \neq F_0 \cap F \subset \beta cl_{\omega_t}(V) \cap F$, $\beta cl_{\omega_t}(V) \cap F \neq \emptyset$ for every $V \in (r, s)\beta O(X, x)$ and every $F \in \mathcal{F}$. So $x \in \beta_{(\omega_r,s)}$-$\theta_t$-$ad\mathcal{F}$.

(c)⇒(e). Let $\mathcal{G}$ be a grill on X. Then $\mathcal{F}(\mathcal{G}) = \{F \subset X : F \cap E \neq \emptyset, \forall E \in \mathcal{G}\}$ is a filter on X and there exists an ultrafilter $\mathcal{F}$ such that $\mathcal{F}(\mathcal{G}) \subset \mathcal{F} \subset \mathcal{G}$ [29]. Let $\mathcal{F}$ β-(ω_r, s)-θ_t-converges to $x \in X$. If possible, let $\mathcal{G}$ does not β-(ω_r, s)-θ_t-converge to x. Then there exists $V \in (r, s)\beta O(X, x)$ such that $E \not\subset \beta cl_{\omega_t}(V)$ and so $E \cap (X - \beta cl_{\omega_t}(V)) \neq \emptyset$ for all $E \in \mathcal{G}$. So $X - \beta cl_{\omega_t}(V) \in \mathcal{F}(\mathcal{G}) \subset \mathcal{G}$. Again by Theorem 4.7, $\beta cl_{\omega_t}(V) \cap F \neq \emptyset$ for every $F \in \mathcal{F}$. So $\beta cl_{\omega_t}(V) \in \mathcal{F}$. Hence $\beta cl_{\omega_t}(V) \in \mathcal{G}$ — a contradiction.

(e)⇒(c). Since every ultrafilter base is a grill, (c) immediately follows.

(d)⇒(b). Let $\{V_\alpha : \alpha \in \Delta\}$ be a family of (r,s)-β-closed subsets of X such that $\cap\{V_\alpha : \alpha \in \Delta\} = \emptyset$. If possible, let $\cap_{\lambda\in\Gamma}\beta int_{\omega_t}(V_\lambda) \neq \emptyset$ for each finite subset Γ of Δ. Then the family $\mathcal{F} = \{\cap\{\beta int_{\omega_t}(V_\gamma), \gamma \in \Gamma\}, \Gamma \subset \Delta, card(\Gamma) < \mathcal{N}_0\}$, where $\mathcal{N}_0$ is the cardinality of the set of all natural numbers is a filter base on X. Then by (d), $\mathcal{F}$ β-(ω_r, s)-θ_t-accumulates at some point x of X. Since $\{X - V_\alpha : \alpha \in \Delta\}$ is a (r,s)-β-open cover of X, $x \in X - V_{\alpha_0}$ for some $\alpha_0 \in \Delta$. Let $G = X - V_{\alpha_0}$. Then $G \in (r,s)\beta O(X,x)$ and $\beta int_{\omega_t}(V_{\alpha_0}) \in \mathcal{F}$ such that $\beta cl_{\omega_t}(G) \cap \beta int_{\omega_t}(V_{\alpha_0}) = \emptyset$ — a contradiction. □

From the above theorem the following theorem follows immediately.

Theorem 4.9. *For a topological space X, the following conditions are equivalent:*
(a) X is β_ω-closed,
(b) for every family $\{V_\alpha : \alpha \in \Delta\}$ of β-closed subsets such that $\cap\{V_\alpha : \alpha \in \Delta\} = \emptyset$, there exist $\alpha_1, \alpha_2, ..., \alpha_n \in \Delta$ such that $\cap_{i=1}^n \beta int_\omega(V_{\alpha_i}) = \emptyset$,
(c) every ultrafilter base β-ω-θ-converges to some point of X,
(d) every filter base β-ω-θ-adheres at some point of X,
(e) every grill on X β-ω-θ-converges to some point of X.

Definition 4.10. A bitopological space X is called β-$(\omega_r, s)t$-regular if for each $x \in X$ and $U \in (r,s)\beta O(X,x)$, there exists a $V \in (r,s)\beta O(X,x)$ such that $cl_{\omega_t}(V) \subset U$. β-$(\omega_r, s)t$-regularity of X is referred as β-ω-regularity if the topologies on X are taken the same.

Theorem 4.11. *For a β-$(\omega_r, s)t$-regular bitopological space X, the following two conditions are equivalent:*
(i) X is $\beta^t_{(\omega_r,s)}$-closed,
(ii) every cover of X by β-(ω_r, s)-θ_t-open sets of X has a finite subcover.

Proof. (i)⇒(ii). Let X be $\beta^t_{(\omega_r,s)}$-closed and Σ be any β-(ω_r, s)-θ_t-open cover of X. Then for each $x \in X$, there exists an $U_x \in \Sigma$ containing x and so Theorem 3.6 ensures the existence of a $V_x \in (r,s)\beta O(X,x)$ such that $\beta cl_{\omega_t}(V_x) \subset cl_{\omega_t}(V_x) \subset U_x$. Since $\{V_x : x \in X\}$ is an (r,s)-β-open cover of X, there exist finite number of points $x_1, x_2, ..., x_k \in X$ such that $X = \cup_{i=1}^k \beta cl_{\omega_t}(V_{x_i})$ and so $X = \cup_{i=1}^k U_{x_i}$. Hence $\{U_{x_i} : x \in X, i = 1, 2, ..., k\}$ is a finite subcover of Σ.

(ii)⇒(i). Let X be a $(\omega_r, s)t$-regular bitopological space. Consider Ω be an (r,s)-β-open cover of X. Let $x \in X$ and $x \in U_x$ for some $U_x \in \Omega$. Since X is β-$(\omega_r, s)t$-regular, there exists $V_x \in (r,s)\beta O(X,x)$ satisfying $\beta cl_{\omega_t}(V_x) \subset cl_{\omega_t}(V_x) \subset U_x$. Then by Lemma 3.6 for each $x \in X$, U_x is

β-(ω_r, s)-θ_t-open set of X. Thus by (ii), there exist $x_1, x_2, ..., x_k \in X$ such that $X = \cup_{i=1}^{k} U_{x_i} \subset \cup_{i=1}^{k} \beta cl_{\omega_t}(U_{x_i})$. □

THEOREM 4.12. *For a β-ω-regular topological space X, the following two conditions are equivalent:*
(i) X is β_ω-closed,
(ii) every cover of X by β-ω-θ-open sets of X has a finite subcover.

THEOREM 4.13. *For a β-$(\omega_r, s)t$-regular bitopological space X, the following two conditions are equivalent:*
(i) X is $\beta^t_{(\omega_r,s)}$-closed,
(ii) every family of β-(ω_r, s)-θ_t-closed subsets of X with finite intersection property has a nonempty intersection.

Proof. (i)⇒(ii). Let X be $\beta^t_{(\omega_r,s)}$-closed and $\{V_\alpha : \alpha \in \Delta\}$ be a family of β-(ω_r, s)-θ_t-closed subsets of X with finite intersection property having empty intersection. Then $\{X - V_\alpha : \alpha \in \Delta\}$ be a β-(ω_r, s)-θ_t-open cover of X. Then the theorem 4.11. ensures the existence of a finite subset Δ_0 of Δ such that $\cup\{X - V_\alpha : \alpha \in \Delta_0\} = X$ and so $\cap\{V_\alpha : \alpha \in \Delta_0\} = \emptyset$ — a contradiction.

(ii)⇒(i). Let X be a β-$(\omega_r, s)t$-regular bitopological space. If X be not $\beta^t_{(\omega_r,s)}$-closed, Theorem 4.11 ensures the existence of a β-(ω_r, s)-θ_t-open cover $\{U_\alpha : \alpha \in \Delta\}$ of X without any finite subcover. Then $\{X - U_\alpha : \alpha \in \Delta\}$ be a family of β-(ω_r, s)-θ_t-closed subsets of X with finite intersection property. Therefore by the hypothesis (ii), $\cap\{X - U_\alpha : \alpha \in \Delta\} \neq \emptyset$ and so $\cup\{U_\alpha : \alpha \in \Delta\} \neq X$ — a contradiction. □

As an application of Theorem 4.13, we prove the following "fixed set theorem" for multifunction:

THEOREM 4.14. *Let X be a $\beta^t_{(\omega_r,s)}$-closed bitopological space and $\Omega : X \to X$ be any multifunction preserving β-(ω_r, s)-θ_t-closed subsets to β-(ω_r, s)-θ_t-closed subsets. Then there exists a subset $K \subset X$ such that $\Omega(K) = K$.*

Proof. It is obvious to note that $\Pi = \{A \subset X : \Omega(A) \subset A, A \neq \emptyset, A$ be β-(ω_r, s)-θ_t-closed$\}$ be a totally order set under the set inclusion. Theorem 3.7.(d) implies that every subfamily of Π has a lower bound and hance Zorns Lemma implies that there is a minimal element K of Π. Since the multifunction Ω preserves β-(ω_r, s)-θ_t-closed subsets to β-(ω_r, s)-θ_t-closed subsets and K is minimal, $K \subset \Omega(K) \subset K$ and hance $\Omega(K) = K$. □

Definition 4.15. A point $x \in X$ is called β-(ω_r, s)-θ_t-complete adherent point of a subset A of X if for each β-(ω_r, s)-θ_t-open set U containing x, $card(A \cap U) = card(A)$. The term β-(ω_r, s)-θ_t-complete adherent point is referred as β-ω-θ-complete adherent point if the topologies on X are taken same.

Definition 4.16. A net $(x_\lambda)_{\lambda \in \Upsilon}$ (where Υ is a directed set) on a bitopological space X is called β-(ω_r, s)-θ_t-adheres at a point $x \in X$ if for every $U \in (r, s)\beta O(X, x)$ and for every $\lambda \in \Upsilon$, there exists $\mu(\succeq \lambda) \in \Upsilon$ such that $x_\mu \in \beta cl_{\omega_t}(U)$. The term β-(ω_r, s)-θ_t-adherent point is referred as β-ω-θ-adhere point if the topologies on X are taken same.

Theorem 4.17. *For a β-$(\omega_r, s)t$-regular bitopological space X, the following three statements are equivalent:*
(i) X is $\beta^t_{(\omega_r,s)}$-closed,
(ii) every net $(x_\lambda)_{\lambda \in \Upsilon}$, where Υ is a well-ordered index set β-(ω_r, s)-θ_t-adheres at a point in X,
(iii) every infinite subset A of X has a β-(ω_r, s)-θ_t-complete adherent point in X.

Proof. (i)⇒(ii). Let X be $\beta^t_{(\omega_r,s)}$-closed and $(x_\lambda)_{\lambda \in \Upsilon}$, where Υ is a well-ordered index set be a net on X. If possible, let $(x_\lambda)_{\lambda \in \Upsilon}$ does not β-(ω_r, s)-θ_t-adhere at any point of X. So for each $x \in X$, there exists an $U_x \in (r, s)\beta O(X, x)$ and a $\lambda(x) \in \Upsilon$ such that $x_\mu \notin \beta cl_{\omega_t}(U_x)$ for all $\mu(\succeq \lambda(x)) \in \Upsilon$. Since $\{V_x : x \in X\}$ forms a cover of X by (r, s)-β-open sets of X, there exist finite number of points $x_1, x_2, ..., x_k \in X$ such that $X = \cup_{i=1}^k \beta cl_{\omega_t}(U_{x_i})$. Consider an $\eta \in \Upsilon$ such that $\eta \succeq \lambda(x_i)$ for all $i = 1, 2, ..., n$. Then for each $i = 1, 2, ..., n$, $x_\mu \notin \beta cl_{\omega_t}(U_{x_i})$ for all $\mu \succeq \eta$ — a contradiction.

(ii)⇒(iii). Let A be an infinite subset of X. Then A can be well-ordered by some minimal well-ordering $\preceq$. Thus A may be thought as a net with a well-ordered index set as domain. Then by (ii), the net A β-(ω_r, s)-θ_t-adheres at a point $x \in X$. Now consider an $U \in (r, s)\beta O(X, x)$. Since X is β-$(\omega_r, s)t$-regular, there exists $V \in (r, s)\beta O(X, x)$ such that $\beta cl_{\omega_t}(V) \subset cl_{\omega_t}(V) \subset U$. Now since net A β-(ω_r, s)-θ_t-adheres at a point $x \in X$, so for any $\lambda \in A$, there exists $\mu(\succeq \lambda) \in A$ such that $x_\mu \in \beta cl_{\omega_t}(V) \cap A$ and hence $card(A) = card(A \cap \beta cl_{\omega_t}(V))$ and for similar cause $card(A \cap U) = card(A \cap \beta cl_{\omega_t}(V))$. Therefore $card(A \cap U) = card(A)$. Hence x is a β-(ω_r, s)-θ_t-complete adherent point of A.

(iii)⇒(i). Let X be not $\beta^t_{(\omega_r,s)}$-closed. Then Theorem 4.11 implies that X has a cover Σ by β-(ω_r, s)-θ_t-open sets of X without any finite subcover. Let ρ be the minimum of the cardinal numbers of the subcover

Σ_0 of Σ. Then clearly $\rho \geq \aleph_0$. Let Σ_0 be well-ordered by minimal well-ordering $\prec$. Then for each $U \in \Sigma_0$, $card(\{W \in \Sigma : W \prec U\}) < \rho$ and so $\{W \in \Sigma : W \prec U\}$ can not be a cover of X. Then for each $U \in \Sigma_0$, there exists $x_U \in X - \cup\{W \in \Sigma : W \prec U\}$. Now consider $P = \{x_U : U \in \Sigma_0\}$. Since $U, V \in \Sigma_0, U \neq V$ implies $x_U \neq x_V$, $card(P) = \rho$. So P is an infinite set. Now consider any $x \in X$. Then $x \in U_0$ for some $U_0 \in \Sigma_0$. Since $x_U \in U_0$ implies $U \prec U_0$. Therefore $\{U \in \Sigma_0 : x_U \in U_0\} \subset \{U \in \Sigma_0 : U \prec U_0\}$ and so by the minimality of $\prec$, we get $card(\{U \in \Sigma_0 : x_U \in U_0\}) < \rho$. Thus $card(A \cap U_0) < \rho = card(A)$. So x can not be β-(ω_r, s)-θ_t-complete adherent point of A. $\square$

THEOREM 4.18. *For a β-$(\omega_r, s)t$-regular topological space X, the following three statements are equivalent:*
(i) X is β_ω-closed,
(ii) every net $(x_\lambda)_{\lambda\in\Upsilon}$, where Υ is a well-ordered index set β-ω-θ-adheres at a point in X,
(iii) every infinite subset A of X has a β-ω-θ-complete adherent point in X.

5. $\beta^t_{(\omega_r,s)}$-closed spaces modulo grill

DEFINITION 5.1. A bitopological space X is called $\beta^t_{(\omega_r,s)}$-closed (resp. H-$\beta^t_{(\omega_r,s)}$-closed) modulo a grill $\mathcal{G}$ if every (r,s)-β-open cover $\{V_\alpha : \alpha \in \Delta\}$ of X has finite subfamily $\{V_{\alpha_i} : \alpha_i \in \Delta, i = 1, 2, ..., n\}$ such that $X - \cup_{i=1}^{n} \beta cl_{\omega_t}(V_{\alpha_i}) \notin \mathcal{G}$ (resp. $X - \cup_{i=1}^{n} cl_{\omega_t}(V_{\alpha_i}) \notin \mathcal{G}$). If the topologies on X are taken the same, then $\beta^t_{(\omega_r,s)}$-closedness (resp. H-$\beta^t_{(\omega_r,s)}$-closedness) modulo a grill $\mathcal{G}$ are referred as β_ω-closed (resp. H-β_ω-closed) modulo the grill $\mathcal{G}$ respectively.

REMARK 5.2. Every $\beta^t_{(\omega_r,s)}$-closed modulo any grill $\mathcal{G}$ bitopological space X is $\beta^t_{(r,s)}$-closed, $P^t_{(\omega_r,s)}$-closedness [2] and H-$\beta^t_{(\omega_r,s)}$-closed modulo the grill $\mathcal{G}$ on X.

THEOREM 5.3. *Every $\beta^t_{(\omega_r,s)}$-closed (resp. $\beta^t_{(r,s)}$-closed and H-$\beta^t_{(\omega_r,s)}$-closed) bitopological space X is $\beta^t_{(\omega_r,s)}$-closed (resp. $\beta^t_{(r,s)}$-closed and H-$\beta^t_{(\omega_r,s)}$-closed) modulo any grill $\mathcal{G}$ on X.*

Proof. We prove the theorem for only $\beta^t_{(\omega_r,s)}$-closedness. Other two are analogous. Let X be $\beta^t_{(\omega_r,s)}$-closed and $\{U_\alpha : \alpha \in \Delta\}$ be any (r,s)-β-open cover of X. Then there exists a finite subset Δ_0 of Δ such that

$X = \cup\{\beta cl_{\omega_t}(U_\alpha) : \alpha \in \Delta_0\}$. Since $\emptyset \notin \mathcal{G}$, $X - \cup\{\beta cl_{\omega_t}(U_\alpha) : \alpha \in \Delta_0\} \notin \mathcal{G}$. $\square$

REMARK 5.4. It is obvious to note that if $\mathcal{G}$ is the grill of all nonempty subsets of any topological space X, then the concepts of X being $\beta^t_{(\omega_r,s)}$-closed (resp. H-$\beta^t_{(\omega_r,s)}$-closed) and $\beta^t_{(\omega_r,s)}$-closed (resp. H-$\beta^t_{(\omega_r,s)}$-closed) modulo the grill $\mathcal{G}$ are equivalent.

THEOREM 5.5. *Let $\mathcal{G}$ be a grill on a topological space X containing all nonempty ω_t-open sets and X is H-$\beta^t_{(\omega_r,s)}$-closed modulo the grill $\mathcal{G}$. Then X is H-$\beta^t_{(\omega_r,s)}$-closed.*

Proof. Let X be H-$\beta^t_{(\omega_r,s)}$-closed modulo the grill $\mathcal{G}$ and $\{U_\alpha : \alpha \in \Delta\}$ be a cover of X by the (r,s)-β-open sets of X. Then there exist finite number of indices $\alpha_1, \alpha_2, ..., \alpha_n \in \Delta$ such that $X - \cup_{i=1}^n cl_{\omega_t}(U_{\alpha_i}) \notin \mathcal{G}$. If $int_{\omega_t}(X - \cup_{i=1}^n (U_{\alpha_i})) \neq \emptyset$, then $int_{\omega_t}(X - \cup_{i=1}^n (U_{\alpha_i})) \in \mathcal{G}$. But $int_{\omega_t}(X - \cup_{i=1}^n (U_{\alpha_i})) = X - cl_{\omega_t}(\cup_{i=1}^n U_{\alpha_i}) = X - \cup_{i=1}^n cl_{\omega_t}(U_{\alpha_i})$. So $X - \cup_{i=1}^n cl_{\omega_t}(U_{\alpha_i}) \in \mathcal{G}$, — a contradiction. Hence $\emptyset = int_{\omega_t}(X - \cup_{i=1}^n (U_{\alpha_i})) = X - \cup_{i=1}^n cl_{\omega_t}(U_{\alpha_i})$. Thus $X = \cup_{i=1}^n cl_{\omega_t}(U_{\alpha_i})$. So X is H-$\beta^t_{(\omega_r,s)}$-closed. $\square$

DEFINITION 5.6. A topological space X is called weakly $\beta^t_{(\omega_r,s)}$-closed (resp. strongly $\beta^t_{(\omega_r,s)}$-closed, strongly (r,s)-β-compact) modulo a grill $\mathcal{G}$ if every (r,s)-β-open (resp. (r,s)-open, (r,s)-β-open) cover $\{V_\alpha : \alpha \in \Delta\}$ of X has finite subfamily $\{V_{\alpha_i} : \alpha \in \Delta, i = 1,2,...,n\}$ such that $X - \cup_{i=1}^n int_{\omega_t}(V_{\alpha_i}) \notin \mathcal{G}$ (resp. $X - \cup_{i=1}^n \beta cl_{\omega_t}(V_{\alpha_i}) \notin \mathcal{G}$, $X - \cup_{i=1}^n V_{\alpha_i} \notin \mathcal{G}$). If the topologies on X are taken the same, then weakly $\beta^t_{(\omega_r,s)}$-closed (resp. strongly $\beta^t_{(\omega_r,s)}$-closed, strongly (r,s)-β-compact) modulo a grill $\mathcal{G}$ are referred as weakly β-ω-closed (resp. strongly β-ω-closed, strongly β-compact) modulo a grill $\mathcal{G}$ modulo the grill $\mathcal{G}$.

DEFINITION 5.7. A topological space X is called strongly $\beta^t_{(\omega_r,s)}$-regular modulo a grill $\mathcal{G}$ if for each $x \in X$ and (r,s)-β-closed set F not containing x there exist disjoint sets $U \in (r,s)\beta O(X,x)$ and $V \in \beta\omega_t O(X)$ such that $F - V \notin \mathcal{G}$. If the topologies on X are taken as the same, the space strongly $\beta^t_{(\omega_r,s)}$-regular modulo a grill $\mathcal{G}$ is referred as strongly β-ω-regular modulo the grill $\mathcal{G}$

THEOREM 5.8. *A $\beta^t_{(\omega_r,s)}$-closed strongly $\beta^t_{(\omega_r,s)}$-regular bitopological space modulo a grill $\mathcal{G}$ is strongly (r,s)-β-compact modulo the grill $\mathcal{G}$.*

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ is a cover of X by (r,s)-β-open sets of X. Then for each $x \in X$, there exists $\alpha(x) \in \Delta$ such that $x \in U_{\alpha(x)}$. Since X is strongly $\beta^t_{(\omega_r,s)}$-regular modulo the grill $\mathcal{G}$, there exist disjoint sets $P_{\alpha(x)} \in (r,s)\beta O(X,x)$ and $Q_{\alpha(x)} \in \beta\omega_t O(X)$ such that $(X - U_{\alpha(x)}) - Q_{\alpha(x)} \notin \mathcal{G}$. Here $\{P_{\alpha(x)} : x \in X\}$ is a cover of X by (r,s)-β-open sets of X. Since X is $\beta^t_{(\omega_r,s)}$-closed, there exist $x(1), x(2), ..., x(n) \in X$ such that $X = \cup_{i=1}^n \beta cl_{\omega_t}(P_{\alpha(x(i))})$. Consider $S_{\alpha(x)} = (X - U_{\alpha(x)}) - Q_{\alpha(x)}$. Here $P_{\alpha(x)} \cap Q_{\alpha(x)} = \emptyset$ implies that $\beta cl_{\omega_t}(P_{\alpha_{(x)}}) \cap Q_{\alpha(x)} = \emptyset$. Now we claim that $\beta cl_{\omega_t}(P_{\alpha(x)}) \subset S_{\alpha(x)} \cup U_{\alpha(x)}$. In fact $q \in \beta cl_{\omega_t}(P_{\alpha_{(x)}})$ but $q \notin U_{\alpha(x)}$ implies that $q \in X - Q_{\alpha(x)}$ and so $q \in ((X - U_{\alpha(x)}) - Q_{\alpha(x)}) = S_{\alpha(x)}$. Thus $X = \cup_{i=1}^n \beta cl_{\omega_t}(P_{\alpha(x(i))}) \subset \cup_{i=1}^n (S_{\alpha(x(i))} \cup U_{\alpha(x(i))})$ and so $X - \cup_{i=1}^n U_{\alpha(x(i))} \subset \cup_{i=1}^n (S_{\alpha(x(i))})$. But for each $i = 1, 2, ..., n$, $S_{\alpha(x(i))} \notin \mathcal{G}$ and so $X - \cup_{i=1}^n U_{\alpha(x(i))} \notin \mathcal{G}$. Hence X is strongly (r,s)-β-compact modulo the grill $\mathcal{G}$. □

THEOREM 5.9. *A β_ω-closed strongly β-ω-regular topological space modulo a grill $\mathcal{G}$ is strongly β-compact modulo the grill $\mathcal{G}$.*

THEOREM 5.10. *A T_2 weakly β_ω-closed space (X,τ) modulo a grill $\mathcal{G}$ is strongly β-ω-regular modulo the grill $\mathcal{G}$.*

Proof. Consider $x \in X$ and β-closed set F not containing x. Then for each $y \in F$, there exist disjoint open sets U_y and V_y containing x and y respectively. Therefore $\{V_y : y \in F\} \cup \{X - F\}$ is a β-open cover of X. Since X is weakly β_ω-closed space modulo the grill $\mathcal{G}$, there exist $y_1, y_2, ..., y_n \in F$ such that $X - [\cup_{i=1}^n int_\omega(V_{y_i}) \cup int_\omega(X - F))] \notin \mathcal{G}$. Now consider $U = X - \beta cl(\cup_{i=1}^n (V_{y_i}))$ and $V = \cup_{i=1}^n (V_{y_i})$. Then $U \cap V = \emptyset$, $U \in \beta O(X,x)$, $V \in \beta O(X) \subset \beta\omega O(X)$ and $F - V = F \cap (X - V) = X - [\cup_{i=1}^n (V_{y_i}) \cup (X - F))] \subset X - [\cup_{i=1}^n int_\omega(V_{y_i}) \cup int_\omega(X - F))]$ and so $F - V \notin \mathcal{G}$. Hence X is strongly β-ω-regular modulo the grill $\mathcal{G}$. □

Let $\mathcal{G}$ be a grill on a topological space (X,τ) and $\phi : P(X) \to P(X)$ be a mapping defined by $\phi(A) = \{x \in X : U \cap A \in \mathcal{G}, \text{ for all } U \in \tau(x)\}$. B. Roy and M. N. Mukherjee [10] has proved that $\psi : P(X) \to P(X)$, where $\psi(A) = A \cup \phi(A)$ for all $A \in P(X)$, is a Kuratowski closure operator and hence induces a topology τ_G on X finer than τ.

THEOREM 5.11. *Let $\mathcal{G}$ be a grill on a topological space (X,τ) and X is strongly β-compact modulo the grill $\mathcal{G}$. Then $(X,\tau_{\mathcal{G}})$ is strongly β-ω-closed modulo the grill $\mathcal{G}$.*

Proof. Let X be strongly β-compact modulo the grill $\mathcal{G}$ and consider $\sum$ be a cover of X by open sets of $(X,\tau_{\mathcal{G}})$. Then for each $x \in X$, there exists $U_x \in \sum$ such that $x \in U_x$. Then there exist a $B_x \in \tau$ and a $V_x \notin \mathcal{G}$ such that $x \in B_x - V_x \subset U_x$. Then $\{B_x : x \in X\}$ is cover of X by open (and so β-open) sets of the space (X, τ). Since (X, τ) is strongly β-compact modulo the grill $\mathcal{G}$, there exist $x(1), x(2), ..., x(n) \in X$ such that $X - \cup_{i=1}^{n} B_{x(i)} \notin \mathcal{G}$. Now $X - \cup_{i=1}^{n} \beta cl_{\omega}^{\tau_{\mathcal{G}}}(U_{x(i)}) \subset X - \cup_{i=1}^{n} U_{x(i)} \subset X - \cup_{i=1}^{n}(B_{x(i)} - V_{x(i)}) \subset (X - \cup_{i=1}^{n}(B_{x(i)})) \cup (\cup_{i=1}^{n}(V_{x(i)})) \notin \mathcal{G}$. Hence $(X, \tau_{\mathcal{G}})$ is strongly β-ω-closed modulo the grill $\mathcal{G}$. □

THEOREM 5.12. *Let $\mathcal{G}$ be a grill on X. A bitopological space X is $\beta^{t}_{(\omega_r,s)}$-closed with respect to the grill $\mathcal{G}$ if and only if every β-(ω_r, s)-θ_t-closed subset of X is $\beta^{t}_{(\omega_r,s)}$-closed with respect to the grill $\mathcal{G}$ and X.*

Proof. Let X be $\beta^{t}_{(\omega_r,s)}$-closed with respect to the grill $\mathcal{G}$ and A be a β-(ω_r, s)-θ_t-closed subset of X and let $\Sigma = \{V_\alpha : \alpha \in \Delta\}$ is a cover of A by (r, s)-β-open sets of X. Since $X - A$ is a β-(ω_r, s)-θ_t-open set, for each $x \in X - A$, by the Lemma 3.6, there exists $U_x \in (r, s)\beta O(X, x)$ such that $\beta cl_{\omega_t}(U_x) \subset X - A$. Hence $\Sigma \cup \{U_x : x \in X - A\}$ is a (r, s)-β-open cover of X and so there exist $\alpha_1, \alpha_2, ..., \alpha_n \in \Delta$ and $x_1, x_2, ..., x_m \in X - A$ such that $X - ((\cup_{i=1}^{n} \beta cl_{\omega_t}(V_{\alpha_i})) \cup (\cup_{i=1}^{m} \beta cl_{\omega_t}(U_{x_i}))) \notin \mathcal{G}$. So $A - \cup_{i=1}^{n} \beta cl_{\omega_t}(V_{\alpha_i}) = A - ((\cup_{i=1}^{n} \beta cl_{\omega_t}(V_{\alpha_i})) \cup (X - A))) \subset A - ((\cup_{i=1}^{n} \beta cl_{\omega_t}(V_{\alpha_i})) \cup (\cup_{i=1}^{m} \beta cl_{\omega_t}(U_{x_i}))) \subset X - ((\cup_{i=1}^{n} \beta cl_{\omega_t}(V_{\alpha_i})) \cup (\cup_{i=1}^{m} \beta cl_{\omega_t}(U_{x_i}))) \notin \mathcal{G}$. Therefore $A - \cup_{i=1}^{n} \beta cl_{\omega_t}(V_{\alpha_i}) \notin \mathcal{G}$ and hence A is $\beta^{t}_{(\omega_r,s)}$-closed with respect to the grill $\mathcal{G}$ and X. Again since X is β-(ω_r, s)-θ_t-closed subset of X, the converse part of the theorem is obvious. □

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